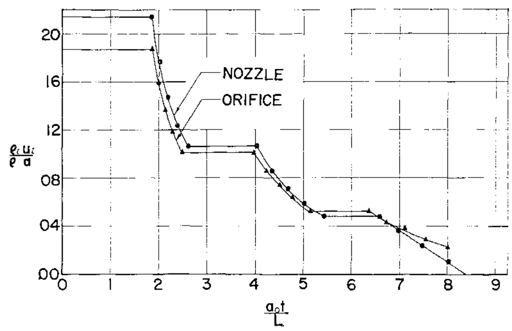

Fig 2 The dimensionless factoı, $\rho_i u_i/\rho_0 a_0$, vs nondimensional time

maining in the vessel is obtained from the pressure relationship, Eqs (4) and (6), and the isentropic relationship:

$$m/m_0 = (P/P_0)^{1/k} \tag{8}$$

Introducing the velocity of approach factor obtained by Kestin and Glass[4] to the quasi-steady results for the discharge through a nozzle, and assuming a coefficient of discharge of one, the modified quasi-steady mass-time curve is plotted in Fig 3 For the sonic portion of the discharge process the velocity of approach factor was calculated continuousl), whereas an average value was used for the subsonic process For the discharge process through an orifice, the quasi-steady approach which includes the velocity of approach factor is fairly difficult to apply due to the large variation of the discharge coefficient during the process However, for the sonic portion of the process, a fairly good representation of the process can be made by assuming a mean value for the discharge coefficient for the pressure range covered, 0 80 for the example under consideration (See Fig 7, Ref 5) The results are plotted in Fig 3

The quasi-steady discharge rate through the nozzle, slope of the mass-time curve (Fig 3), is plotted in Fig 4 for a comparison with the results based on wave theory

Discussion of Results and Conclusions

A comparison of the results (Fig 3) indicates that the mass-time relationship for the sonic discharge of a finite vessel through a nozzle can be closely approximated by the application of the velocity of approach correction factor to the quasi-steady results For the sonic discharge through an orifice the results depend on the estimate of the average discharge coefficient for the process In a subsonic dischaıge process the pressure in the vessel may fall appreciably below the pressure of the surrounding medium which the quasi-steady approach fails to predict In such cases there may be discrepancies between the quasi-steady and wave theory results In addition, the quasi-steady results predict a continuously changing mass flow rate, whereas wave theory predicts periods of constant mass discharge (Fig 4)

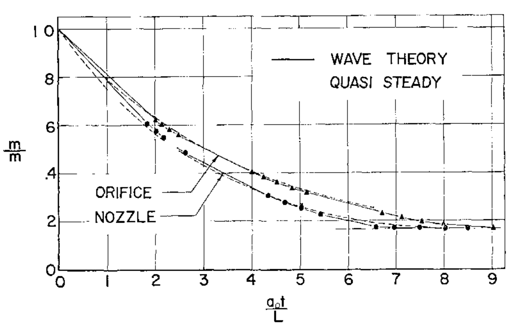

Fig 3 A comparison of the quasi-steady and wave theoıy results of mass vs time

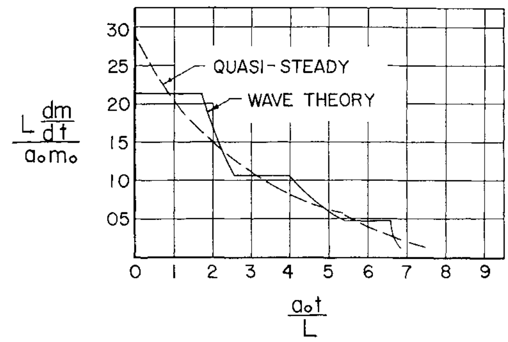

Fig 4 A compaıison of the quasi-steady and wave theoıy mass dischaıge rates for the dischaıge thıough a nozzle

In conclusion, the deteımination of the mass and mass discharge rate for a finite vessel can be calculated from the wave diagrams of the method of characteristics The modified quasi-stead) technique of Kestin and Glass can be applied to the discharge of a finite vessel, and will closely approximate the mass based on wave theory for the sonic portion of the discharge process if the appropriate discharge coefficient is used

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Buckling Load of Bars with Variable Stiffness: A Simple Numerical Method

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THERE are occasions when the buckling load of a stepped bar, or a bar with varying moment of inertia along its length, is desired Classical methods are, no doubt, available, but all of them involve elaborate computations and/or theoıy The method presented here is simple and direct It is based on the finite differences technique in a way

The governing differential equation for a buckled bar with hinged ends is well known It is

$$EIY'' + PY = 0$$

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Table 1 Determination of critical load [$I = I_0(1 + \sin\pi x/L)$]

Location	1	2	3	4	5	6	7	Remarks
I/I_0	1	1 5000	1 8660	2 0000	1 8660	1 5000	1	
Y	0	1 0000	0 5000	0 5000	1 5000	2 5000	0	Initial set
K/I_0		−2 2500	1 8660	4 0000	0	−2 1000		Av K/I_0 =0 3032
Y	0	0 2270	0 3362	0 8534	1 5507	0 7042	0	
K/I_0		−0 7788	2 2647	0 4220	−1 8577	0 3032		Av K/I_0 = 0 0706
Y	0	0 1642	0 4993	1 0072	0 8398	0 4102	0	
K/I_0		1 5608	0 6455	−1 3409	−0 5824	0 0706		Av K/I_0 = 0 0707
Y	0	0 2439	0 6139	0 7142	0 5518	0 2695	0	
K/I_0		0 7753	−0 1897	−0 7357	−0 4051	0 0707		Av K/I_0 = −0 2229
Y	0	0 3316	0 5561	0 5866	0 4553	0 2459	0	
K/I_0		−0 4843	−0 6510	−0 5518	−0 3197	−0 2229		Av K/I_0 = −0 4459
Y	0	0 3266	0 5186	0 5480	0 4508	0 2648	0	
K/I_0		−0 6183	−0 5848	−0 4621	−0 3678	−0 4459		Av K/I_0 = −0 4958
Y	0	0 3106	0 4951	0 5399	0 4640	0 2779	0	
K/I_0		−0 6092	−0 5264	−0 4471	−0 4429	−0 4958		Av K/I_0 = −0 5043
Y	0	0 2976	0 4842	0 5425	0 4743	0 2850	0	
K/I_0		−0 5594	−0 4944	−0 4663	−0 4762	−0 5043		Av K/I_0 = −0 5001
Y	0	0 2905	0 4809	0 5459	0 4798	0 2879	0	
K/I_0		−0 5167	−0 4869	−0 4800	−0 4892	−0 5001		Av K/I_0 = −0 4946
Y	0	0 2879	0 4806	0 5479	0 4878	0 2884	0	
K/I_0		−0 4964	−0 4866	−0 4873	−0 4924	−0 4946		Av K/I_0 = −0 4914
Y	0	0 2874	0 4810	0 5488	0 4821	0 2883	0	
K/I_0		−0 4894	−0 4881	−0 4901	−0 4920	−0 4914		Av K/I_0 = −0 4902
Y	0	0 2875	0 4814	0 5490	0 4820	0 2880	0	
K/I_0		−0 4882	−0 4893	−0 4907	−0 4911	−0 4902		Av K/I_0 = −0 4899
Y	0	0 2876	0 4816	0 5490	0 4818	0 2879	0	
K/I_0		−0 4889	−0 4900	−0 4906	−0 4905	−0 4899		Av K/I_0 = −0 4900

Same average K/I_0 obtained in the next iteration

with the usual notations Let the bar be divided into n parts The differential equation is satisfied at each of these points dividing the bar

Therefore, at the ith point, we may write, using central differences technique, an approximation to Y'' as follows:

$$Y'' = \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{h^2}$$

where h is the spacing between the points Thus, the differential equation at each i is

$$EI_i \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{h^2} + PY_i = 0$$

$$Y_{i+1} + Y_{i-1} - 2Y_i = -\frac{h^2PY_i}{EI_i} = \frac{K_iY_i}{I_i}$$

where

$$K_i = -\frac{h^2P}{E}$$

Therefore,

$$I_i(Y_{i-1} + Y_{i+1} - 2Y_i) = K_iY_i \tag{1}$$

and

$$\frac{Y_{i-1} + Y_{i+1}}{2 + (K_i/I_i)} = Y_i \tag{2}$$

The method suggested herein makes use of Eqs (1) and (2) The method will be explained by means of an example Let the moment of inertia of a bar whose buckling load is desired vary as $I(x) = I_0(1 + \sin\pi x/L)$ This problem is picked from Refs 1 and 2 Let us arbitrarily divide the bar into six segments and mark the stations 1 through 7 as shown in Table 1 The table illustrates the step-by-step procedure and will aid the discussion here

1) The moment of inertia at each station is written below it A set of completely arbitrary displacements is given at each of these stations and written below each station as shown

2) By use of Eq (1) K_i is calculated for each i An average of these K_i is calculated and entered as shown

3) Now, the new set of displacements is calculated from Eq (2), using K_{av} at each point The newly computed Y_{i-1} may be used at i to accelerate the process

4) Steps 2 and 3 are repeated until the average K does not differ from the previous value to within a small error (without too much difference among K_i)

5) P_c is determined from this K

Comments and Conclusions

Table 1 shows the quickness with which the process converges A critical load of 17 6468 EI_0/L^2 is obtained after 16 iterations with $h = L/6$ As the calculations in the table show, the iterations were continued until there was no difference between the successive K_{av}s

Misse,[1] using Raleigh's principle, established a bound for the critical load of this bar which was shown to be 17 9034 $EI_0/L^2 \leq P_{cr} \leq$ 18 0613 EI_0/L^2 By this method P_{cr} was found to be equal to 17 929 EI_0/L^2, after 16 iterations with $h = L/10$ It was observed[3] that the number of iterations were as many as 16 because the initial set of displacements was purposely assumed to be a rather unreasonable set

It is believed that this method is useful in such problems and eliminates long and laborious calculations The computations involved are simple and could be carried on a slide rule or desk calculator

Work is under way to extend this technique to columns with other boundary conditions and to the determination of P_c for higher modes

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